A Bayesian latent process spatiotemporal regression model for areal count data

Supplementary material

1 Moment properties of the general model

Here, we provide the moment properties of the general form of the proposed models in Section 2 of the main paper, since other models can be obtained from it by making appropriate substitutions for the $A$ matrix. We first derive the variance-covariance matrix and lag-$h$ autocovariance of the latent process, $X_t$, since these are needed to derive the unconditional moments of the observed counts, $Y_t$. It is necessary to note that since $X_t$ is multivariate Gaussian, its second moment involves the variances of its individual components as well as the covariances between them. This is not the case with $Y_t$ whose components are mutually independent Poisson random variables.

Noting that $X_t$ is a second-order stationary process, it can be shown that

$$
\Sigma_X = A \Sigma_X A' + \tau_\xi^{-1} 11' + \Sigma,
$$

(1)

where $\Sigma_X$ denotes the variance-covariance matrix of $X_t$. A useful approach for solving for $\Sigma_X$ in (1) is to vectorize the equation (Lütkepohl, 2005, p.27). This yields

$$
\text{vec}(\Sigma_X) = (A \otimes A) \text{vec}(\Sigma_X) + \text{vec}(\tau_\xi^{-1} 11' + \Sigma)
= (I_{n^2} - A \otimes A)^{-1} \text{vec}(\tau_\xi^{-1} 11' + \Sigma),
$$

(2)

where ‘$\otimes$’ denotes the Kronecker product. Equation (2) shows that the (non-diagonal) transition matrix $A$ contributes significantly to the covariances between the components of $X_t$ through $(I_{n^2} - A \otimes A)^{-1}$. This implies that $A$ accounts for significant spatial interdependence in $X_t$ in the model. To obtain the lag-$h$ autocovariance of $X_t$ (i.e. $\Gamma_X(h)$), we post-multiply its
evolution equation in equation (5) of the main paper by $X'_{t-h}$ and take expectation to obtain

$$E[X_tX'_{t-h}] = E[AX_{t-1}X'_{t-h}] + 1E[\xi_tX'_{t-h}] + E[\epsilon_tX'_{t-h}].$$ \hspace{1cm} (3)

Since $\epsilon_t$ and $\xi_t$ are each uncorrelated with $X_{t-h}$ for $h > 0$, we obtain that

$$\Gamma_X(h) = A\Gamma_X(h-1) = A^h\Gamma_X(0).$$ \hspace{1cm} (4)

Equation (4) can be computed easily when $A$, $\tau_\xi$ and $\Sigma$ are known since $\Gamma_X(0) = \Sigma_X$. Note that due to the dependence of $X_{t,i}$ on $X_{t-1}$, it is often easier to obtain its marginal moments from the moments of $X_t$ as has been outlined.

To derive the unconditional moments of $Y_t$, we note again that even though the $X_{t,i}$’s are spatially dependent, by definition, $Y_{t,1}, \ldots, Y_{t,n}$ are spatially independent, each having an independent Poisson distribution conditional on the corresponding $X_{t,i}$. Hence, only the diagonal elements of $\Sigma_X$ and $\Gamma_X(h)$ are needed to obtain the unconditional moments of $Y_t$. For notational convenience, conciseness and to facilitate direct reference to the first level of the hierarchical models in the main paper, we obtain and give the unconditional moments of $Y_{t,i}$ ($t = 1, \ldots, T; \ i = 1, \ldots, n$) as follows.

**Unconditional Mean of $Y_{t,i}$:**

$$\mu_{Y_{t,i}} = E[Y_{t,i}] = E[E[Y_{t,i}|X_{t,i}]] = E_{t,i}\exp\left(z'_{t,i}\beta_i + \frac{\Sigma_{X,i}}{2}\right),$$ \hspace{1cm} (5)

where $\Sigma_{X,i}$ is the $i$th diagonal element of $\Sigma_X$.

**Unconditional Variance of $Y_{t,i}$:**

$$Var[Y_{t,i}] = \mu_{Y_{t,i}} + \mu_{Y_{t,i}}^2(e^{\Sigma_{X,i}} - 1).$$ \hspace{1cm} (6)

**Lag-$h$ Autocovariance of $Y_{t,i}$:**

$$Cov[Y_{t,i}, Y_{t+h,i}] = \mu_{Y_{t,i}}\mu_{Y_{t+h,i}}(e^{\Gamma_X(i,h)} - 1), \quad \text{for} \ h > 0;$$ \hspace{1cm} (7)
where $\Gamma_{X,i}(h)$ is the $i$th diagonal element of $\Gamma_X(h)$.

**Lag-$h$ Autocorrelation of $Y_{t,i}$:**

$$Cor[Y_{t,i}, Y_{t+h,i}] = \frac{e^{\Gamma_{X,i}(h)} - 1}{\sqrt{(\mu_{Y_{t,i}}^{-1} + (e^{\Sigma X_{i}} - 1))(\mu_{Y_{t+h,i}}^{-1} + (e^{\Sigma X_{i}} - 1))}}$$  \hspace{1cm} (8)

These results are easy to obtain using conditional expectation results and the properties of the lognormal and Poisson distributions. We see from (6) that $\text{Var}[Y_{t,i}] > \mu_{Y_{t,i}}$, with the overdispersion term dominated by the moments of $X_{t,i}$. This shows that the latter introduces overdispersion in the observed counts. These results further demonstrate how the spatial dynamics inherited by $X_{t,i}$ from its dependence on $X_{t-1,i}$ and $X_{t-1,j}, j \neq i$ is transferred to $Y_{t,i}$ through the conditional specification in the proposed models. The autocorrelation function of $Y_{t,i}$ can seen to be a function of the moments of the underlying $X_{t,i}$ and since it can be shown that $Cor[Y_{t,i}, Y_{t+h,i}] < \Gamma_{X,i}(h)$ (see [Davis et al., 2000]), the autocorrelation in the observed counts cannot be greater than that of the underlying latent process.

**References**
